My interest in this note is to give some examples of eigenvalues of a matrix:

Example 1: Let us choose A to be the matrix

$$A = \begin{pmatrix} 2 & 4 & 5 \\ 3 & 6 & 7 \\ -1 & 4 & -3 \end{pmatrix}$$

The characteristic polynomial of A is obtained by computing

$$det(\lambda I - A) = \lambda^3 - 5\lambda^2 - 47\lambda - 6$$

The roots of the above polynomial are at 9.8389, -4.7094 and -0.1295 which are the three eigenvalues of the matrix A. Note that all three eigenvalues are real.

Example 2: Let us choose A to be the matrix

$$A = \begin{pmatrix} 5 & 0 & 1 \\ 8 & 3 & 6 \\ 0 & -6 & 3 \end{pmatrix}$$

The characteristic polynomial of A is obtained by computing

$$det(\lambda I - A) = \lambda^3 - 11\lambda^2 + 75\lambda - 177$$

The roots of the above polynomial are at $3.6581 + 5.8879i$, $3.6581 - 5.8879i$ and 3.6838 which are the three eigenvalues of the matrix A. Note that all the eigenvalues are not real, but the eigenvalues occur in complex conjugate pairs.

Let us now look at a sufficiently complicated example:

Example 3: Let us choose A to be the matrix

$$A = \begin{pmatrix} 5 & 0 & 1 & 1 & 0 & 0 \\ 8 & 8 & 6 & 0 & 1 & 0 \\ 0 & -6 & 8 & 0 & 0 & 1 \\ 1 & 0 & 0 & 5 & 0 & 1 \\ 0 & 0 & 0 & 8 & 3 & 6 \\ 0 & -1 & 0 & 3 & -6 & 3 \end{pmatrix}$$

The characteristic polynomial of A is obtained by computing

$$det(\lambda I - A) = \lambda^6 - 32\lambda^5 + 482\lambda^4 - 4078\lambda^3 + 21054\lambda^2 - 60643\lambda + 7$$

The roots of the above polynomial are at $8.4927 + 6.3631i$, $8.4927 - 6.3631i$, $3.6075 + 5.5606i$, $3.6075 - 5.5606i$, 4.9295 and 2.8701 which are the six eigenvalues of the matrix A. Note that all the eigenvalues are not real, but the eigenvalues occur in complex conjugate pairs.

Remark: The above three examples are examples of matrices that have distinct eigenvalues but the eigenvalues are not necessarily real. This forces us to learn ‘complex arithmetic’. Since sooner or later, we would need to learn about complex numbers any way — why not today?

What are complex numbers?

If a and b are two real numbers, then a complex number is a number that looks like $a + ib$. a is called the *real part* and b is called the *imaginary part* of the complex number. Complex numbers can be added, subtracted, multiplied and divided, although some of these operations may be a bit awkward. My job is to make you feel comfortable with these operations.

If $a_1 + ib_1$, $a_2 + ib_2$ are two complex numbers then we define

$$(a_1 + ib_1) + (a_2 + ib_2) = ((a_1 + a_2) + i(b_1 + b_2))$$

$$(a_1 + ib_1) - (a_2 + ib_2) = ((a_1 - a_2) + i(b_1 - b_2))$$

$$(a_1 + ib_1) \times (a_2 + ib_2) = ((a_1a_2 - b_1b_2) + i(a_1b_2 + a_2b_1))$$

$$(a_1 + ib_1) \div (a_2 + ib_2) = \left(\frac{a_1a_2 + b_1b_2}{a_2^2 + b_2^2} + i\frac{a_2b_1 - a_1b_2}{a_2^2 + b_2^2}\right), \text{provided } (a_2, b_2) \neq (0, 0).$$

The quadratic power of $a + ib$ is given as follows:

$$(a + ib)^2 = a^2 + i^2b^2 + i2ab = (a^2 - b^2) + i(2ab)$$

where we need to remember that $i^2 = -1$.

How to calculate power of a complex numbers?

What if we want to calculate $(a + ib)^{98}$? It will be somewhat awkward to multiply $a + ib$ ninety eight times. One therefore uses what is called the polar coordinates.

Let us define $r = \sqrt{a^2 + b^2}$. Let us also define an angle θ such that

$$cos\theta = \frac{a}{r} \ , \ sin\theta = \frac{b}{r}.$$

It is easy to see that $a + ib = r(cos\theta + isin\theta)$. It follows that

$$(a + ib)^n = [r(cos\theta + isin\theta)]^n$$

One can show that

$$(cos\theta + i \ sin\theta)^n = (cos \ n\theta + i \ sin \ n\theta)$$

which is also called the DeMoivre's Theorem. In particular, note that

$$(cos\theta + i \ sin\theta)^2 = (cos^2\theta - sin^2\theta) + i(2sin\theta \ cos\theta) = (cos \ 2\theta + i \ sin \ 2\theta).$$

The general formula is easily shown using trigonometry but we shall not do it here. We however have the following magical formula:

$$(a + ib)^n = [r^n(cos \ n\theta + i \ sin \ n\theta)]$$

which is how we calculate power of a complex number.

For example $(3.6581 - 5.8879i)^{98}$ is calculated as follows:

$$r = \sqrt{3.6581^2 + 5.8879^2} = 6.9317$$

$$cos\theta = .527735, \ sin\theta = -.8494164.$$

It follows that $\theta = -58.14825$ degrees.

$$cos \ 98\theta = .4776, \ sin \ 98\theta = .8786.$$

We conclude that $(3.6581 - 5.8879i)^{98} = 6.9317^{98}(.4776 \ + \ i \ .8786)$.

How to calculate exponential of a complex numbers?

We want to calculate
$$e^{a+ib}.$$
Of course, we know that
$$e^{(x+y)} = e^x \ e^y,$$
where x and y are scalar real numbers. We extend this property and write
$$e^{a+ib} = e^a e^{ib}.$$
We now define $e^{ib} = cos \ b \ + \ i \ sin \ b$. It follows that
$$e^{\ a+ib} = e^a \ cos \ b + ie^a \ sin \ b.$$
As an example, if we want to calculate $e^{3.6581-i5.8879}$, we have
$$e^{3.6581-i5.8879} = e^{3.6581}(cos5.8879 - isin5.8879) = e^{3.6581}(.9229 + i.3851) = 38.7876(.9229 + i.3851)$$
and we have
$$e^{3.6581-i5.8879} = 35.7971 + 14.9371.$$

A vector of complex numbers.

Many often, we are interested in looking at a vector of complex numbers. For example $\mathbb{C}^2$ is a pair of complex numbers. Elements of $\mathbb{C}^2$ are of the form:
$$u \ = \ \begin{pmatrix} a + b \ i \\ c + d \ i \end{pmatrix}$$
where a, b, c and d are real numbers. The magnitude $\|u\|$ is given by $\sqrt{a^2 + b^2 + c^2 + d^2}$. The dot product is defined as follows. Assume that
$$v \ = \ \begin{pmatrix} e + f \ i \\ g + h \ i \end{pmatrix},$$
we define
$$u \ . \ v \ = \ (a + b \ i)(e - f \ i) + (c + d \ i)(g - h \ i).$$
Note that $u \ . \ v \neq v \ . \ u$, rather $u \ . \ v = \overline{v \ . \ u}$, where ‘overline’ stands for conjugation. If $z = a + ib$, we define the conjugate $\bar{z} = a - ib$.

Two complex vectors u and v are called perpendicular or orthogonal if $u \ . \ v \ = \ 0$.

It can be easily seen that $\|u\| = \sqrt{u \ . \ u}$.

We now go back to our story of matrix A.

Example 1 (continued): Recall that the matrix A was given by

$$A = \begin{pmatrix} 2 & 4 & 5 \\ 3 & 6 & 7 \\ -1 & 4 & -3 \end{pmatrix}.$$

The Cayley Hamilton Theorem tells us that

$$A^3 - 5A^2 - 47A - 6I = 0,$$

which implies that

$$A^3 = 5A^2 + 47A + 6I.$$

It would follow that

$$A^4 = 72A^2 + 241A + 30I.$$

In general, an arbitrary power A^n of A for $n \geq 3$ can be written as a linear combination of A^2, A and I. The process of writing may actually be quite tedious. Hence we resort to the following trick:

Write

$$A^n = \alpha + \beta A + \gamma A^2,$$

where α, β and γ are computed by replacing A by the eigenvalues of A in the above equation. Assume that the eigenvalues are at $\lambda_1 = 9.8389$, $\lambda_2 = -4.7094$ and $\lambda_3 = -0.1295$. We obtain the following equation:

$$\begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 \\ 1 & \lambda_2 & \lambda_2^2 \\ 1 & \lambda_3 & \lambda_3^2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} \lambda_1^n \\ \lambda_2^n \\ \lambda_3^n \end{pmatrix}.$$

Solving the above equation, we obtain

$$\alpha = 0.0042\lambda_1^n - 0.0191\lambda_2^n + 1.0149\lambda_3^n,$$

$$\beta = 0.0334\lambda_1^n - 0.1457\lambda_2^n + 0.1124\lambda_3^n,$$

and

$$\gamma = 0.0069\lambda_1^n + 0.0150\lambda_2^n - 0.0219\lambda_3^n.$$

It follows that

$$A^n = (0.0042\lambda_1^n - 0.0191\lambda_2^n + 1.0149\lambda_3^n) + (0.0334\lambda_1^n - 0.1457\lambda_2^n + 0.1124\lambda_3^n)A + (0.0069\lambda_1^n + 0.0150\lambda_2^n - 0.0219\lambda_3^n)A^2.$$

We remark that the above calculation illustrates the power of eigenvalues.

Example 2 (continued) In this example the matrix A is given by

$$A = \begin{pmatrix} 5 & 0 & 1 \\ 8 & 3 & 6 \\ 0 & -6 & 3 \end{pmatrix}.$$

The Cayley Hamilton Theorem tells us that

$$A^3 - 11A^2 + 75A - 177I = 0.$$

It follows that

$$A^3 = 11A^2 - 75A + 177I.$$

The eigenvalues of the matrix A are at

$$\lambda_1 = r(cos\theta + isin\theta),$$

$$\lambda_2 = r(cos\theta - isin\theta)$$

and at a real value λ_3. Proceeding as before, we write

$$A^n = \alpha + \beta A + \gamma A^2,$$

where the coefficients α, β and γ are computed by solving the equations

$$\begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 \\ 1 & \lambda_2 & \lambda_2^2 \\ 1 & \lambda_3 & \lambda_3^2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} \lambda_1^n \\ \lambda_2^n \\ \lambda_3^n \end{pmatrix}.$$

In the above matrix we have complex entries. Collecting the real and the imaginary parts, we obtain the following:

$$\begin{pmatrix} 1 & rcos\theta & r^2cos2\theta \\ 0 & rsin\theta & r^2sin2\theta \\ 1 & \lambda_3 & \lambda_3^2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} r^n cos\ n\theta \\ r^n sin\ n\theta \\ \lambda_3^n \end{pmatrix}.$$

The above equation contain only real entries. Thus, even when the eigenvalues are complex conjugates, a real solution can still be obtained. This procedure is illustrated in this example.

We will soon be talking about eigenvectors of a matrix. However, before we do that, let us make some points.

Point Unus: If A is a symmetric matrix, then all its eigenvalues are real.

A symmetric matrix, by definition, is a matrix A such that $A = A^T$, i.e. a symmetric matrix is equal to its own transpose. Let us look at some examples:

Example 4:

Here are the matlab codes:
```
A=[2 4 5;3 6 7;-1 4 -3]
A=A+A'
poly(A)
roots(poly(A))
```

We are looking at the symmetric matrix A given by

$$A = \begin{pmatrix} 4 & 7 & 4 \\ 7 & 12 & 11 \\ 4 & 11 & -6 \end{pmatrix}.$$

The characteristic polynomial of the matrix A is given by

$$det(\lambda I - A) = \lambda^3 - 10\lambda^2 - 234\lambda + 54.$$

The roots of the characteristic polynomial are real and are given by

21.0134, –11.2420, 0.2286.

Example 5:

We are looking at the symmetric matrix A given by

$$A = \begin{pmatrix} 3 & 3 & 0 & 0 & 0 & 0 \\ 3 & 6 & 0 & 1 & 0 & 0 \\ 0 & 0 & 6 & 0 & 2 & 0 \\ 0 & 1 & 0 & 5 & 3 & 0 \\ 0 & 0 & 2 & 3 & 6 & 1 \\ 0 & 0 & 0 & 0 & 1 & 9 \end{pmatrix}.$$

The characteristic polynomial of the matrix A is given by

$$det(\lambda I - A) = \lambda^6 - 35\lambda^5 + 477\lambda^4 - 3167\lambda^3 + 10415\lambda^2 - 15195\lambda + 7470.$$

The roots of the characteristic polynomial are real and are given by

9.9794, 8.5917, 7.8124, 5.6143, 2.0178, 0.9844.

Note that in this example the eigenvalues are all real, as is expected because the matrix A is symmetric. But also note that additionally all the eigenvalues are positive.

Point Duo:

If A is a symmetric matrix, then it is called *positive definite* **if all its eigenvalues are real and positive.**

If A is a symmetric matrix, then it is called *positive semidefinite* **if all its eigenvalues are real and non negative.**

If A is a symmetric matrix, then it is called *negative definite* **if all its eigenvalues are real and negative.**

If A is a symmetric matrix, then it is called *negative semidefinite* **if all its eigenvalues are real and non positive.**

In example 5, the matrix A is positive definite, symmetric matrix. In example 4, the matrix A is neither positive definite nor negative definite. (You may call it an indefinite matrix).

There is a beautiful theorem about positive definite symmetric matrices that might be worth knowing. It goes like this:

Let A be any $n \times n$ symmetric matrix. Let us define matrices A_j by choosing the first j rows and j columns from the matrix A, for $j = 1, \cdots n$.

Point Tres:

A symmetric matrix A is positive definite if and only if

$$det(A_j) > 0, \quad j = 1, \cdots n.$$

Example 4 (continued):

$$A_1 = \begin{pmatrix} 4 \end{pmatrix}.$$

$$A_2 = \begin{pmatrix} 4 & 7 \\ 7 & 12 \end{pmatrix}.$$

$$A_3 = \begin{pmatrix} 4 & 7 & 4 \\ 7 & 12 & 11 \\ 4 & 11 & -6 \end{pmatrix}.$$

Calculating the determinants, we obtain $det(A_1) = 4 > 0$, $det(A_2) = -1 < 0$. Hence the matrix A is not positive definite. The point I am making is that we can test positive definiteness without calculating the roots of the characteristic polynomial (a somewhat difficult step back in those days when root finding programs were not available on a laptop.

Example 5 (continued):

$$A_1 = \begin{pmatrix} 3 \end{pmatrix}, \quad \det(A_1) = 3.$$

$$A_2 = \begin{pmatrix} 3 & 3 \\ 3 & 6 \end{pmatrix}, \quad \det(A_2) = 9.$$

$$A_3 = \begin{pmatrix} 3 & 3 & 0 \\ 3 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix}, \quad \det(A_3) = 54.$$

$$A_4 = \begin{pmatrix} 3 & 3 & 0 & 0 \\ 3 & 6 & 0 & 1 \\ 0 & 0 & 6 & 0 \\ 0 & 1 & 0 & 5 \end{pmatrix}, \quad \det(A_4) = 252.$$

$$A_5 = \begin{pmatrix} 3 & 3 & 0 & 0 & 0 \\ 3 & 6 & 0 & 1 & 0 \\ 0 & 0 & 6 & 0 & 2 \\ 0 & 1 & 0 & 5 & 3 \\ 0 & 0 & 2 & 3 & 6 \end{pmatrix}, \quad \det(A_5) = 858.$$

$$A_6 = \begin{pmatrix} 3 & 3 & 0 & 0 & 0 & 0 \\ 3 & 6 & 0 & 1 & 0 & 0 \\ 0 & 0 & 6 & 0 & 2 & 0 \\ 0 & 1 & 0 & 5 & 3 & 0 \\ 0 & 0 & 2 & 3 & 6 & 1 \\ 0 & 0 & 0 & 0 & 1 & 9 \end{pmatrix}, \quad \det(A_6) = 7470.$$

The matrix A is thus positive definite.

To summarize, what we have learnt today are the following:

1) Matrices have eigenvalues and that these eigenvalues are important in calculating powers of a matrix.
2) Eigenvalues, in general, can be real or complex and therefore we need to deal with complex numbers.
3) We need to calculate powers of complex numbers. DeMoivre's theorem comes in handy.
4) All eigenvalues of a symmetric matrix are real.
5) All eigenvalues of a symmetric, positive definite matrix are real and positive.
6) There is a beautiful test for positive definiteness of a symmetric matrix using determinants of minors.

That is a lot for one day, isn't it.